Classification of Möbius Maps

Hyperbolic and Loxodromic Maps, Reflections, and Parabolic Maps

Lecture Notes

October 22, 2026

1 Fixed Points and the Trace Criterion

Let T be a Möbius map, $T \neq \mathrm{id}$. We look for its **fixed points** (possibly ∞):

$$z = \frac{az+b}{cz+d} \iff cz^2 + (d-a)z - b = 0.$$

Normalize so that $ad - bc = 1$.

Case count

Case 1 (two fixed points): $(d-a)^2 \neq -4bc \iff (a+d)^2 \neq 4$.

Case 2 (one fixed point, "parabolic"): $(d-a)^2 = -4bc \iff a+d = \operatorname{Tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm 2$.

2 Case 1: Two Fixed Points

Let z_1, z_2 be the fixed points; start with the normalized case $z_1 = 0$, $z_2 = \infty$. Then $c = 0$, $b = 0$, $ad = 1$, and

$$Tz = \frac{a}{d} z = a^2 z.$$

Hyperbolic and Loxodromic Maps

Case 1(b): $a^2 > 0$, $a^2 \neq 1$ – hyperbolic. Write $Tz = kz$, $k > 0$, $k \neq 1$: a dilation. It leaves each ray $L_\varphi = \{\arg z = \varphi\}$ invariant, and shifts C_r to C_{kr}.

$$\operatorname{Tr} T = \pm\Big(\sqrt{k} + \frac{1}{\sqrt{k}}\Big) \in (-\infty, -2) \cup (2, \infty).$$

Case 1(c): $a^2 \notin \mathbb{R}_+ \cup \{|a^2| = 1\}$ – loxodromic. A composition of an elliptic and a hyperbolic map with the same fixed points; $\operatorname{Tr} T \notin \mathbb{R}$.

Arbitrary fixed points z_1, z_2. Let $Sz := \dfrac{z - z_1}{z - z_2}$, so $Sz_1 = 0$, $Sz_2 = \infty$, and set $\widetilde{T} := STS^{-1}$. Then

$$\widetilde{T}(\infty) = ST(z_2) = S(z_2) = \infty, \qquad \widetilde{T}(0) = ST(z_1) = S(z_1) = 0,$$

so $\widetilde{T}$ is exactly T in normalized position, and $\operatorname{Tr}\widetilde{T} = \operatorname{Tr} T$ (conjugate matrices have equal trace).

Definition

T is **elliptic/hyperbolic/loxodromic** if $\widetilde{T} = STS^{-1}$ is. Equivalently: $\operatorname{Tr} T \in (-2, 2)$ is elliptic; $\operatorname{Tr} T \in (-\infty, -2) \cup (2, \infty)$ is hyperbolic; $\operatorname{Tr} T \notin \mathbb{R}$ is loxodromic. In every case $T = S^{-1}\widetilde{T}S$.

3 Geometry: Apollonius and Steiner Circles

Definition

$$C_r(z_1, z_2) := \{|Sz| = r\} = \left\{\frac{|z - z_1|}{|z - z_2|} = r\right\} \quad \text{– \textbf{circles of Apollonius},}$$

$$L_\varphi(z_1, z_2) := \{\arg(Sz) = \varphi\} \quad \text{– circular arcs from } z_1 \text{ to } z_2 \text{ (\textbf{Steiner circles}).}$$

Reflections

With respect to $C_r(z_1, z_2)$: fixes each $L_\varphi(z_1, z_2)$ setwise, and maps $C_s(z_1, z_2)$ to $C_{r^2/s}(z_1, z_2)$.
With respect to $L_\varphi(z_1, z_2)$: fixes each $C_r(z_1, z_2)$ setwise, and maps $L_\theta(z_1, z_2)$ to $L_{2\varphi-\theta}(z_1, z_2)$.

Proof. True for $z_1 = 0, z_2 = \infty$: reflection in $C_r = \{|z| = r\}$ is $z \mapsto r^2/\bar{z}$, which fixes each ray $L_\varphi = \{\arg z = \varphi\}$ (as $\arg(r^2/\bar{z}) = \arg z$) and sends $C_s = \{|z| = s\}$ to $\{|z| = r^2/s\}$; reflection in the line L_φ through the origin is $z \mapsto e^{2i\varphi}\bar{z}$, which fixes each C_r (as $|e^{2i\varphi}\bar{z}| = |z|$) and sends L_θ to $L_{2\varphi-\theta}$ (as $\arg(e^{2i\varphi}\bar{z}) = 2\varphi - \arg z$). The general case follows since S is Möbius and so preserves symmetry with respect to any circle or line. □

4 Case 2: One Fixed Point (Parabolic)

Here $\operatorname{Tr} T = \pm 2$, $T \neq \mathrm{id}$, and T has a unique fixed point z_0.

Subcase $z_0 = \infty$. Then $Tz = z + b$ for some $b \neq 0$ (a translation). Define, for $w \in \mathbb{C}$,

$$V_w^b := \{w + tb : t \in \mathbb{R}\} \quad \text{(line through } w \text{ in direction } b\text{)},$$
$$H_w^b := \{w + itb : t \in \mathbb{R}\} \quad \text{(perpendicular line)}.$$

These two families are mutually perpendicular, $V_w^b \perp H_w^b$, and

$$TV_w^b = V_w^b, \qquad TH_w^b = H_{w+b}^b :$$

T fixes each line parallel to b (setwise) and shifts each perpendicular line to the next one, by b.

Subcase $z_0 \neq \infty$. Let $Sz := \dfrac{1}{z - z_0}$ (sending $z_0 \to \infty$), and $\widetilde{T} := STS^{-1}$, so $\widetilde{T}(\infty) = \infty$ with no other fixed point, $\operatorname{Tr} T = \operatorname{Tr} \widetilde{T} = \pm 2$, and $T = S^{-1}\widetilde{T}S$.

Definition (horocycles)

$$V_w^b(z_0) := S^{-1}V_w^b, \qquad H_w^b(z_0) := S^{-1}H_w^b :$$

circles through z_0, all mutually tangent at z_0 (since the lines V_w^b, resp. H_w^b, are all asymptotically parallel, i.e. tangent to each other at ∞, and S^{-1} preserves tangency). These are called **horocycles**.

As before,

$$TV_w^b(z_0) = V_w^b(z_0), \qquad TH_w^b(z_0) = H_{w+b}^b(z_0),$$

and reflections behave just as nicely as in the elliptic/hyperbolic case.

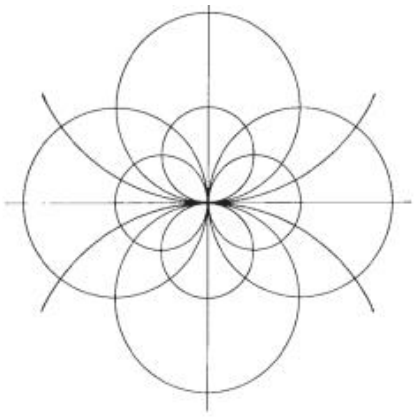


Figure 1: The horocycles $V_w^b(z_0)$, $H_w^b(z_0)$: two mutually orthogonal families of circles, all tangent to each other at the fixed point z_0 (the image of the parallel/perpendicular line families under S^{-1}).



August Ferdinand Möbius (1790–1868)